tangens (E') plano (x, z) secatur in recta, cuius aequatio est $mx'x - z'z - f^2 = 0$, et quae ipsa versus axem XX' inclinata est angulo ψ eo, ut sit $\operatorname{tg}\psi = \frac{mx'}{z'}$. Jam si ω significat angulum inter proiectionem (II) et axem XX' interceptum, simul esse debet et $\operatorname{tg}\omega = -\frac{a''}{b''}$ et $\operatorname{tg}\omega = -\frac{1}{\operatorname{tg}\psi}$, itaque $\frac{a'}{b'} = \frac{z'}{mx'}$. Inde sequitur, quaesitae normalis aequationes esse has:

$$y - y' - \frac{ny'}{mx'}(x - x') = 0,\quad z - z' + \frac{z'}{mx'}(x - x') = 0.$$

31. *Problema. Datis puncti alicuius (P) superficiei S coordinatis x', y', z' determinare binos illi puncto respondentes radios curvaturae superficiei S.*

Solutio. Constat, unicuique puncto superficiei alicuius curvae duos respondere radios curvaturae illius superficiei expressos aequatione hac: $\varrho = (z'' - z')\sqrt{\left(1 + \left(\frac{dz'}{dx'}\right)^2 + \left(\frac{dz'}{dy'}\right)^2\right)}$, ubi z'' una est coordinatarum, quae determinant centrum curvaturae, ipsa vero quantitas $(z'' - z') = \delta$ determinatur aequatione:

$$(N)\ldots\left\{\begin{aligned} &\left(\left(\frac{d^2z'}{dx'^2}\right)\left(\frac{d^2z'}{dy'^2}\right) - \left(\frac{d^2z'}{dx'dy'}\right)\right)\delta^2 \\ &+ \left(2\left(\frac{dz'}{dx'}\right)\left(\frac{dz'}{dy'}\right)\left(\frac{d^2z'}{dx'dy'}\right) - \left(1 + \left(\frac{dz'}{dy'}\right)^2\right)\left(\frac{d^2z'}{dx'^2}\right) - \left(1 + \left(\frac{dz'}{dx'}\right)^2\right)\left(\frac{d^2z'}{dy'^2}\right)\right)\delta \\ &+ 1 + \left(\frac{dz'}{dx'}\right)^2 + \left(\frac{dz'}{dy'}\right) = 0. \end{aligned}\right.$$

(Cf. praeter alios *Brandes* höhere Geometrie Th. II. §. 314). Quae si referuntur ad aequationem (G) superficiei S, reperiuntur haec:

$$\left(\frac{dz'}{dx'}\right) = \frac{mx'}{z'},\ \left(\frac{dz'}{dy'}\right) = \frac{ny'}{z'},\ \left(\frac{d^2z'}{dx'dy'}\right) = -\frac{mnx'y'}{z'^3},$$

$$\left(\frac{d^2z'}{dx'^2}\right) = \frac{m(z'^2 - mx'^2)}{z'^3},\ \left(\frac{d^2z'}{dy'^2}\right) = \frac{n(z'^2 - ny'^2)}{z'^3}.$$

Substitutis his valoribus et soluta aequatione (N) invenitur radii curvaturae ϱ valor hic duplex:

$$\varrho = [-P \pm Q]\,\frac{\sqrt{(m^2x'^2 + n^2y'^2 + z'^2)}}{2mnf^2},$$

ubi brevitatis causa positum est:

$$P = m^2(n-1)\,x'^2 + n^2(m-1)\,y'^2 + (m+n)\,z'^2,$$

$$Q^2 = \left(m^2(n+1)\,x'^2 - n^2(m+1)\,y'^2 - (m-n)\,z'^2\right)^2 + 4m^2n^2(m+1)(n+1)\,x'^2y'^2.$$

Et quum sit $Q^2 - P^2 = 4mn\,[mx'^2 + ny'^2 + z'^2]\,f^2$, quae necessario quantitas est positiva, ita ut semper sit $Q > P$, sequitur inde, alterum radii ϱ valorem semper esse positivum, alterum negativum.