32. Quadratura sive complanatio certae cuiusdam partis superficiei S non nisi per approximationem effici potest. Significet, ut exemplum ponam, F illam superficiei S partem, quae terminatur plano (x, y) et alio aliquo plano, quod illi aequidistet et exprimatur aequatione $z = h$, (partem illam, quae in Fig. 1 intercepta est inter ellipses BLHM et blhm;) tum ad determinandam aream F evadit aequatio haec:

$$F = \int dx \int dy \sqrt{\left(1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right)} = \int dx \int dy \sqrt{\left(\frac{m(m+1)\,x^2 + n(n+1)y^2 - f^2}{mx^2 + ny^2 - f^2}\right)},$$

sive posito $m(m+1))\,x^2 - f^2 = A$, $n(n+1) = B$, $mx^2 - f^2 = C$, $n = D$:

$F = \int dx \int \frac{(A + By^2)\,dy}{\sqrt{(AC + (AD + BC)\,y^2 + BDy^4)}}$. Integranda igitur est functio differentialis huius formae: $\frac{(\alpha + \beta y^2)\,dy}{\sqrt{(a + by^2 + cy^4)}}$, quod sola approximationis via fieri posse constat. Si singulare illud contingit, ut sit $m = n$, superficies S formatur rotatione hyperbolae aequatione $mx^2 - z^2 = f^2$ expressae circa axem eius coniugatum, qui situs est in axe ZZ′, quare tum erit:

$$dF = 2\pi x\,dx \sqrt{\left(1 + \left(\frac{dz}{dx}\right)^2\right)} = \frac{2\pi}{m}\,dz \sqrt{((m+1)z^2 + mf^2)};$$

hinc invenitur integrando intra limites $z = o$ et $z = h$:

$$F = \frac{\pi h}{m} \sqrt{((m+1)\,h^2 + mf^2)} + \frac{\pi f^2}{\sqrt{(m+1)}} \log\left(\frac{h\sqrt{(m+1)} + \sqrt{((m+1)\,h^2 + mf^2)}}{f\sqrt{m}}\right).$$

33. Facilius est determinare quantitatem voluminis a duobus planis plano (x, y) parallelis et superficiei S parte planis illis terminata circumclusi. Unumquodque planum plano (x, y) aequidistans superficiem S secat in ellipsi, cuius aequatio est $mx^2 + ny^2 = f^2 + z^2$ (§. 2), ubi litera z significat distantiam plani secantis a pl. (x, y). Jam si s significat aream ellipseos illius, et K volumen inter planum secans, planum (x, y) et superficiem S interceptum, erit $s = \frac{\pi(f^2 + z^2)}{\sqrt{(mn)}}$ et $dK = \frac{\pi(f^2 + z^2)dz}{\sqrt{(mn)}}$, ergo $K = \frac{\pi z}{\sqrt{(mn)}}\,(f^2 + \tfrac{1}{3} z^2) + C$.

Itaque sumpto integrali intra limites $z = o$ et $z = h$ prodit:

$K\begin{smallmatrix}(h)\\(o)\end{smallmatrix} = \frac{\pi h}{\sqrt{(mn)}}\,(f^2 + \tfrac{1}{3} h^2)$. Inde sequitur $K\begin{smallmatrix}(+h)\\(-h)\end{smallmatrix} = \frac{2\pi h}{\sqrt{(mn)}}\,(f^2 + \tfrac{1}{3} h^2)$,

$K\begin{smallmatrix}(h')\\(h)\end{smallmatrix} = \frac{\pi f^2(h' - h) + \frac{1}{3}\pi(h'^3 - h^3)}{\sqrt{(mn)}}$. Sit V volumen cylindri, cuius bases sint ellipses illae, quae simul solido, cuius volumen est $= K\begin{smallmatrix}(+h)\\(-h)\end{smallmatrix}$, sunt pro basibus, quare eiusdem cylindri altitudo est $= 2h$, et volumen $= V = \frac{2h\pi(f^2 + h^2)}{\sqrt{(mn)}}$; itaque si litera R denotat volumen solidi a cylindri huius superficie curva et superficie S circumclusi, erit